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Stability of damped membranes and plates with distributed inputs

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Abstract

This paper proves the stability of boundary and distributed damped membranes and Kirchhoff plates under distributed inputs. Distributed viscous or Kelvin–Voigt damping ensures a weakly bounded response to a bounded transverse loading for pinned membranes and clamped plates. Damping on part of the boundary can also weakly stabilize the forced response, provided the damped and undamped boundary normals satisfy certain conditions. For example, damping on half and one side of the boundary is sufficient for circular and rectangular domains, respectively.

1. Introduction

Many engineering applications have distributed parameter models governed by partial differential equations. Often forcing of unknown but bounded magnitude disturbs the system and the boundedness of the response needs to be determined. Without damping, flexible structures are not stable due to resonances corresponding to natural frequencies in the system. Bounded sinusoidal inputs at these frequencies cause an unbounded response.

One approach to determine the stability of distributed parameter flexible systems is to discretize using Galerkin, FEM, or finite difference approximations [1]. The system reduces to a set of finite, second-order differential equations with mass, damping, and stiffness matrices. These systems are exponentially (and bounded input–bounded output) stable if the stiffness matrix is positive definite (no rigid body modes) and the damping matrix satisfies complete or pervasive damping conditions [2]. Unfortunately, these conditions only apply to the discretized model, not the full order distributed system.

Recently, researchers have made progress in the stability analysis of distributed parameter systems. Cavalcanti and Oquendo [3] show exponential and polynomial decay for a partially viscoelastic nonlinear wave equation subject to nonlinear and localized frictional damping. Cheng [4] proves the continuity of the input/output map for boundary control systems through the system transfer function. Komornik [5] and Lagnese [6] use the multiplier method to prove the boundary stabilization of membranes and plates. Guesmia [7] provides decay estimates when integral inequalities cannot be applied due to the lack of dissipativity. Zhao

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and Rahn [8] apply the energy multiplier method to damped strings and beams, proving bounded response to distributed inputs.

This paper extends the approach in Ref. [8] to two-dimensional membranes and plates by using the Lyapunov-based energy multiplier method and a variety of integral inequalities [9–12]. Distributed viscous and material damping and boundary damping are shown to weakly stabilize the response to bounded distributed inputs. The energy multiplier method is used because the mathematics are relatively simple (compared to semigroup analysis, for example) and the functionals derived as part of the stability analysis can be used for Lyapunov-based (e.g. adaptive [13], iterative [14], and repetitive [15]) control development.

2. Mathematical preliminaries

Viscoelastic material behavior, frictional interaction between contacting surfaces, or movement through a dissipative fluid cause damping in flexible structures. Distributed (viscous and material) and boundary (viscous) damping are analyzed in this paper. Viscous damping forces are produced when the structure moves through fluid and are proportional to transverse velocity. Kelvin–Voigt damping is due to material viscoelasticity and proportional to material strain rate.

The following equalities and inequalities are used extensively in this paper and are presented without proof (see Refs. [9–12] for details). Throughout the paper, we assume that a two-dimensional open, bounded, connected, Lipschitz domain with boundary Γ is defined.

2.1. Equalities

The divergence theorem applies to vector fields $V = P(x_1, x_2)\mathbf{i} + Q(x_1, x_2)\mathbf{j}$ as follows:

$$\int_\Omega \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}\right) \mathrm{d}x = \int_\Gamma (P\,\mathrm{d}x_2 - Q\,\mathrm{d}x_1). \tag{1}$$

The normal derivative of $w(\mathbf{x}, t)$ is defined as

$$\frac{\partial w}{\partial n} = \nabla w \cdot \mathbf{n} \quad \text{on } \Gamma, \tag{2}$$

where $\mathbf{n}$ is the unit-normal vector to Γ pointing toward the exterior of Ω.

The following integral equalities apply to $w \in H^1(\Omega)$ and $\upsilon \in H^2(\Omega)$:

$$\int_\Omega \Delta\upsilon w\,\mathrm{d}x = \int_\Gamma \frac{\partial \upsilon}{\partial n} w\,\mathrm{d}\Gamma - \int_\Omega \nabla\upsilon \cdot \nabla w\,\mathrm{d}x, \tag{3}$$

$$\int_\Omega \mathbf{r} \cdot \nabla w\,\mathrm{d}x = \int_\Gamma (\mathbf{r} \cdot \mathbf{n}) w\,\mathrm{d}\Gamma - \int_\Omega (\nabla \cdot \mathbf{r}) w\,\mathrm{d}x. \tag{4}$$

The divergence of products can be calculated as follows:

$$\nabla \cdot (w\mathbf{a}) = w\nabla \cdot \mathbf{a} + (\nabla w) \cdot \mathbf{a}, \tag{5}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\nabla \cdot \mathbf{a})\mathbf{b} + (\nabla \cdot \mathbf{b})\mathbf{a}, \tag{6}$$

where $\mathbf{r}$, $\mathbf{a}$, $\mathbf{b}$ are two-dimensional vectors.

2.2. Inequalities

The nonlinear damping inequality is

$$(\mathbf{a} \cdot \mathbf{b}) \leqslant \delta|\mathbf{a}|^2 + \frac{1}{\delta}|\mathbf{b}|^2. \tag{7}$$

The Poincaré inequality

$$\int_{\Omega} w^2 \,\mathrm{d}x \leqslant m_1 \int_{\Omega} |\nabla w|^2 \,\mathrm{d}x \tag{8}$$

holds $\forall w \in H^2(\Omega)$ with $w = 0$ on Γ for some constant $m_1 > 0$. The Sobolev inequality is

$$\int_{\Gamma_1} w^2 \,\mathrm{d}x \leqslant m_2 \int_{\Omega} |\nabla w|^2 \,\mathrm{d}x \quad \forall w \in H^1(\Omega), \tag{9}$$

where m_2 is a positive constant, $\Gamma = \Gamma_0 \cup \Gamma_1$, and $w = 0$ on Γ_0.

3. Damped membranes

For the damped membrane model shown in Fig. 1, we assume that the membrane is inextensible and perfectly flexible, the in-plane stress P is constant, and bounded distributed forcing $f(\mathbf{x}, t)$ is applied in the domain Ω. First, a membrane with distributed viscous and material damping is considered. Then, we consider a membrane without damping in the field equation and with a damped boundary condition on Γ_1 with the remaining boundary Γ_0 pinned.

3.1. Distributed damped membranes

The field equation, boundary conditions, and initial conditions of the damped membrane are:

$$\rho\ddot{w} + b\dot{w} - D\Delta\dot{w} - P\Delta w = f \quad \text{in } \Omega \times R_+, \tag{10}$$

$$w(\mathbf{x}, t) = 0 \quad \text{on } \Gamma \times R_+, \tag{11}$$

$$w(\mathbf{x}, 0) = w_0 \quad \text{on } \Omega, \tag{12}$$

$$\dot{w}(\mathbf{x}, 0) = \dot{w}_0 \quad \text{on } \Omega, \tag{13}$$

where dots indicate time differentiation, ρ is the mass/area, b is viscous damping, D is Kelvin–Voigt damping, Γ is the boundary, Ω is the domain, and $\mathbf{n}$ is the unit-normal vector to Γ pointing toward the exterior of Ω. We assume the models presented in this paper are well-posed and possess a unique solution for all initial conditions and bounded inputs. We prove weak stability or boundedness of the response $w(\mathbf{x}, t)$ to strongly or pointwise bounded disturbances $|f(\mathbf{x}, t)| < M < \infty$, $\forall \mathbf{x} \in \Omega$ and $t > 0$ ($f \in \mathscr{L}_\infty(\Omega)$). This means that we show the energy, a quadratic functional of the distributed displacement and velocity, is bounded ($E(t) \in \mathscr{L}_\infty$). We do not prove, however, that the displacement is pointwise or strongly bounded.

Theorem 1. *The response of the damped membrane governed by Eqs.* (10)–(13) *is weakly bounded if either b or D is nonzero and* $f(\mathbf{x}, t) \in \mathscr{L}_\infty(\Omega)$.

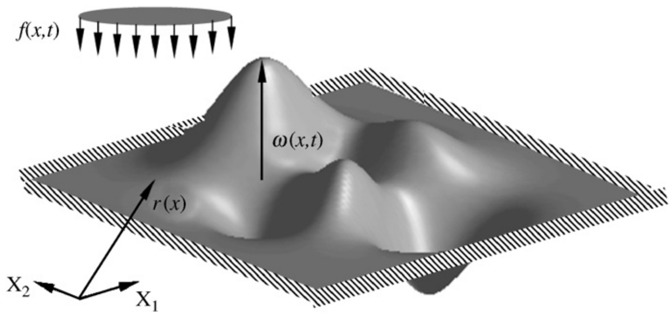

Fig. 1. Schematic diagram of a distributed damped membrane/plate with distributed disturbances.

Proof. The energy of the membrane

$$E = \frac{1}{2}\int_{\Omega}(\rho\dot{w}^2 + P|\nabla w|^2)\,\mathrm{d}x \geqslant 0 \tag{14}$$

has a time rate of change which can be upper bounded by

$$\begin{aligned}
\dot{E} = &\int_{\Omega}[\dot{w}(f - b\dot{w} + D\Delta\dot{w} + P\Delta w) + P\nabla w\cdot\nabla\dot{w}]\,\mathrm{d}x \\
&\leqslant \delta_1\int_{\Omega}\dot{w}^2\,\mathrm{d}x + \frac{1}{\delta_1}\int_{\Omega}f^2\,\mathrm{d}x - b\int_{\Omega}\dot{w}^2\,\mathrm{d}x + D\int_{\Gamma}\frac{\partial\dot{w}}{\partial n}\dot{w}\,\mathrm{d}\Gamma \\
&\quad - D\int_{\Omega}|\nabla\dot{w}|^2\,\mathrm{d}x + P\int_{\Gamma}\frac{\partial w}{\partial n}\dot{w}\,\mathrm{d}\Gamma - P\int_{\Omega}(\nabla w\cdot\nabla\dot{w})\,\mathrm{d}x \\
&\quad + P\int_{\Omega}(\nabla w\cdot\nabla\dot{w})\,\mathrm{d}x \\
&\leqslant -\left(b + \frac{D}{2m_1} - \delta_1\right)\int_{\Omega}\dot{w}^2\,\mathrm{d}x - \frac{D}{2}\int_{\Omega}|\nabla\dot{w}|^2\,\mathrm{d}x + \frac{1}{\delta_1}\int_{\Omega}f^2\,\mathrm{d}x,
\end{aligned}\tag{15}$$

where Eqs. (3), (7), and (8) are used. Inequality of Eq. (15) lacks the $\int_{\Omega}|\nabla w|^2$ term that appears in E. We therefore define a new functional by adding the crossing term $C(t)$

$$V(t) = E(t) + \beta C(t), \tag{16}$$

where $\beta > 0$ and

$$C(t) = \rho\int_{\Omega}\dot{w}w\,\mathrm{d}x. \tag{17}$$

The functional $V(t)$ is positive because

$$\begin{aligned}
|C(t)| &\leqslant \frac{1}{2}\rho\int_{\Omega}(\dot{w}^2 + w^2)\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\rho\int_{\Omega}(\dot{w}^2 + m_1|\nabla w|^2)\,\mathrm{d}x \\
&\leqslant \frac{\rho\max(1, m_1)}{\min(\rho, P)}\frac{1}{2}\int_{\Omega}(\rho\dot{w}^2 + P|\nabla w|^2)\,\mathrm{d}x \\
&= \eta E,
\end{aligned}\tag{18}$$

using inequalities in Eqs. (7) and (8), where

$$\eta = \frac{\rho\max(1, m_1)}{\min(\rho, P)}. \tag{19}$$

This means that

$$0 \leqslant \lambda_1 E(t) \leqslant V(t) \leqslant \lambda_2 E(t), \tag{20}$$

where $\lambda_1 = 1 - \beta\eta > 0$, and $\lambda_2 = 1 + \beta\eta > 1$, for sufficiently small β. Differentiation of the crossing term produces

$$\begin{aligned}
\dot{C} &= \int_{\Omega}\rho\ddot{w}\,w\,\mathrm{d}x + \int_{\Omega}\rho\dot{w}^2\,\mathrm{d}x \\
&= \int_{\Omega}(f - b\dot{w} + D\Delta\dot{w} + P\Delta w)w\,\mathrm{d}x + \rho\int_{\Omega}\dot{w}^2\,\mathrm{d}x \\
&= \rho\int_{\Omega}\dot{w}^2\,\mathrm{d}x + \dot{C}_1 + \dot{C}_2 + \dot{C}_3 + \dot{C}_4.
\end{aligned}\tag{21}$$

The terms in Eq. (21) simplify as follows:

$$\begin{aligned}\dot{C}_1 &= \int_\Omega f w \,\mathrm{d}x \\ &\leqslant \delta_2 \int_\Omega w^2 \,\mathrm{d}x + \frac{1}{\delta_2}\int_\Omega f^2 \,\mathrm{d}x \\ &\leqslant \delta_2 m_1 \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \frac{1}{\delta_2}\int_\Omega f^2 \,\mathrm{d}x,\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{C}_2 &= -\int_\Omega b\dot{w}w \,\mathrm{d}x \\ &\leqslant b\delta_3 \int_\Omega w^2 \,\mathrm{d}x + \frac{b}{\delta_3}\int_\Omega \dot{w}^2 \,\mathrm{d}x \\ &\leqslant b m_1 \delta_3 \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \frac{b}{\delta_3}\int_\Omega \dot{w}^2 \,\mathrm{d}x,\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{C}_3 &= \int_\Omega D w \Delta \dot{w} \,\mathrm{d}x \\ &= D\int_\Gamma \frac{\partial \dot{w}}{\partial n} w \,\mathrm{d}\Gamma - D\int_\Omega (\nabla w \cdot \nabla \dot{w}) \,\mathrm{d}x \\ &\leqslant D\delta_4 \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \frac{D}{\delta_4}\int_\Omega |\nabla \dot{w}|^2 \,\mathrm{d}x,\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{C}_4 &= \int_\Omega P w \Delta w \,\mathrm{d}x \\ &= \int_\Gamma P\frac{\partial w}{\partial n} w \,\mathrm{d}\Gamma - P\int_\Omega |\nabla w|^2 \,\mathrm{d}x \\ &= -P\int_\Omega |\nabla w|^2 \,\mathrm{d}x,\end{aligned} \tag{25}$$

using the boundary condition of Eq. (11) and Eqs. (3), (7), and (8). Substitution of Eqs. (22)–(25) into Eq. (21) yields

$$\begin{aligned}\dot{C} \leqslant &-[P - (\delta_2 + b\delta_3)m_1 - D\delta_4]\int_\Omega |\nabla w|^2 \,\mathrm{d}x + \frac{1}{\delta_2}\int_\Omega f^2 \,\mathrm{d}x \\ &+ \left(\rho + \frac{b}{\delta_3}\right)\int_\Omega \dot{w}^2 \,\mathrm{d}x + \frac{D}{\delta_4}\int_\Omega |\nabla \dot{w}|^2 \,\mathrm{d}x.\end{aligned} \tag{26}$$

Substitution of the derivative of the crossing term of (Eq. (26)) into Eq. (14) produces

$$\begin{aligned}\dot{V} \leqslant &-\left[b + \frac{D}{2m_1} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right)\right]\int_\Omega \dot{w}^2 \,\mathrm{d}x \\ &- \beta[P - (\delta_2 + b\delta_3)m_1 - D\delta_4]\int_\Omega |\nabla w|^2 \,\mathrm{d}x \\ &- D\left(\frac{1}{2} - \frac{\beta}{\delta_4}\right)\int_\Omega |\nabla \dot{w}|^2 \,\mathrm{d}x + \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\int_\Omega f^2 \,\mathrm{d}x \\ \leqslant &-\lambda_3 E + \varepsilon,\end{aligned} \tag{27}$$

where, for sufficiently small β, δ_1, δ_2, δ_3, and δ_4,

$$\frac{1}{2} \geqslant \frac{\beta}{\delta_4}, \tag{28}$$

$$\varepsilon_1 = b + \frac{D}{2m_1} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right) > 0, \tag{29}$$

$$\varepsilon_2 = \beta[P - (\delta_2 + b\delta_3)m_1 - D\delta_4] > 0, \tag{30}$$

$$\varepsilon = \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right) \max_{t\in[0,\infty)} \int_\Omega f^2\, \mathrm{d}x < \infty, \tag{31}$$

$$\lambda_3 = \frac{\min(\varepsilon_1, \varepsilon_2)}{\max(\rho, P)} > 0 \tag{32}$$

for bounded f. By using Eq. (20), we obtain

$$\dot{V} \leqslant -\lambda V + \varepsilon, \tag{33}$$

where $\lambda = \lambda_3/\lambda_2$, with the solution

$$V(t) \leqslant V(0)\mathrm{e}^{-\lambda t} + \frac{\varepsilon}{\lambda} \in \mathscr{L}_\infty \tag{34}$$

and

$$E(t) \leqslant \frac{1}{\lambda_1} V(t) \in \mathscr{L}_\infty. \qquad \square \tag{35}$$

Thus, the system is weakly stable with respect to the energy norm.

3.2. Boundary damped membranes

We remove the distributed damping in Eq. (10) to obtain the two-dimensional wave equation with partially damped boundary conditions shown in Fig. 2. The governing equations are:

$$\rho\ddot{w} - P\Delta w = f \quad \text{in } \Omega \times R_+, \tag{36}$$

$$w = 0 \quad \text{on } \Gamma_0 \times R_+, \tag{37}$$

$$P\frac{\partial w}{\partial n} + c\dot{w} = 0 \quad \text{on } \Gamma_1 \times R_+, \tag{38}$$

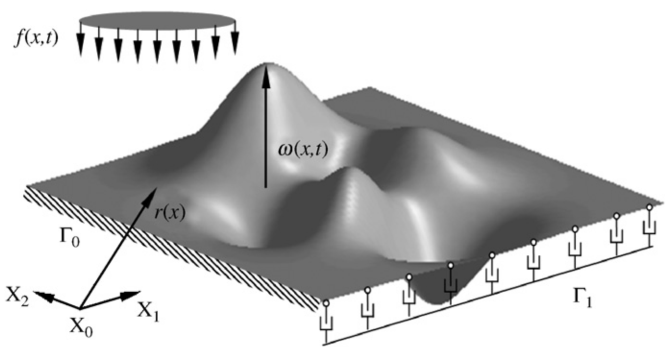

Fig. 2. Schematic diagram of a boundary damped membrane/plate with distributed disturbances.

where $\Gamma = \Gamma_0 \cup \Gamma_1$, c is the boundary viscous damping coefficient, and the initial conditions are given in Eqs. (12) and (13). We assume the boundary normals satisfy

$$\mathbf{r} \cdot \mathbf{n} \leqslant 0 \quad \text{on } \Gamma_0, \tag{39}$$

$$\mathbf{r} \cdot \mathbf{n} > 0 \quad \text{on } \Gamma_1, \tag{40}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{x}_0 \in \mathbf{R}^2$ [5,6].

Theorem 2. *The response of the boundary damped membrane governed by Eqs.* (36)–(38) *is weakly bounded if* $c>0, f \in \mathscr{L}_\infty(\Omega)$, *and the boundary normal conditions given in Eqs.* (39) *and* (40) *are satisfied.*

Proof. The energy given in Eq. (14) has a time derivative

$$\begin{aligned}\dot{E} &= \int_\Omega (\dot{w} f + P\dot{w}\Delta w + P\nabla w \cdot \nabla \dot{w})\,\mathrm{d}x \\ &\leqslant -c\int_{\Gamma_1} \dot{w}^2\,\mathrm{d}\Gamma + \delta_1 \int_\Omega \dot{w}^2\,\mathrm{d}x + \frac{1}{\delta_1}\int_\Omega f^2\,\mathrm{d}x\end{aligned} \tag{41}$$

using Eqs. (3), (7), and (38).

The boundary damper does not match the distributed input, providing neither a negative kinetic nor potential energy domain integral. A positive functional is defined as in Eq. (16) with a different crossing term

$$C(t) = C_1 + C_2, \tag{42}$$

where $C_1 = 2\int_\Omega \rho\dot{w}(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x$ and $C_2 = \int_\Omega \rho\dot{w}w\,\mathrm{d}x$.

We bound the crossing term of Eq. (42) with respect to the system energy as follows:

$$\begin{aligned}|C(t)| &\leqslant 2\rho R\int_\Omega |\dot{w}||\nabla w|\,\mathrm{d}x + \frac{1}{2}\rho\int_\Omega (\dot{w}^2 + w^2)\,\mathrm{d}x \\ &\leqslant \frac{2\rho R}{2}\int_\Omega (\dot{w}^2 + |\nabla w|^2)\,\mathrm{d}x + \frac{1}{2}\rho\int_\Omega (\dot{w}^2 + m_1|\nabla w|^2)\,\mathrm{d}x \\ &\leqslant \eta E,\end{aligned} \tag{43}$$

by using Eqs. (7) and (8), where

$$R = \sup_{\Gamma_1} \|\mathbf{r}(\mathbf{x})\|, \tag{44}$$

$$\eta = \frac{\rho\max[(2R+1),\ (2R+m_1)]}{\min(\rho, P)}. \tag{45}$$

The time derivative of the crossing term of Eq. (42) depends on

$$\begin{aligned}\dot{C}_1 &= 2\int_\Omega (f + P\Delta w)(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x + 2\rho\int_\Omega \dot{w}(\mathbf{r}\cdot\nabla\dot{w})\,\mathrm{d}x \\ &\leqslant 2R\int_\Omega |f||\nabla w|\,\mathrm{d}x + 2P\int_\Omega \Delta w(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x \\ &\quad + 2\rho\int_\Omega \dot{w}(\mathbf{r}\cdot\nabla\dot{w})\,\mathrm{d}x \\ &\leqslant 2R\delta_2\int_\Omega |\nabla w|^2\,\mathrm{d}x + \frac{2R}{\delta_2}\int_\Omega f^2\,\mathrm{d}x + \dot{C}_3 + \dot{C}_4,\end{aligned} \tag{46}$$

where the inequality in Eq. (7) is used.

The third term in Eq. (46) simplifies as follows:

$$
\begin{aligned}
\dot{C}_3 &= 2P\int_\Omega \Delta w(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x \\
&= 2P\int_\Gamma (\mathbf{r}\cdot\mathbf{n})|\nabla w|^2\,\mathrm{d}\Gamma - 2P\int_\Omega |\nabla w|^2\,\mathrm{d}x - P\int_\Omega \mathbf{r}\cdot\nabla(|\nabla w|^2)\,\mathrm{d}x \\
&\leqslant \frac{Rc^2}{P}\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma, \qquad (47)
\end{aligned}
$$

using the boundary conditions (Eqs. (37) and (38), Eqs. (2)–(4), and Eq. (7)). Based on Eq. (2) and the boundary condition of Eq. (39), $2P\int_{\Gamma_0}(\mathbf{r}\cdot\mathbf{n})|\nabla w|^2\,\mathrm{d}\Gamma\leqslant 0$ is dropped from Eq. (47). The fourth term in Eq. (46) can be written as

$$
\begin{aligned}
\dot{C}_4 &= 2\rho\int_\Omega \dot{w}(\mathbf{r}\cdot\nabla\dot{w})\,\mathrm{d}x \\
&\leqslant 2\rho R\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma - 4\rho\int_\Omega \dot{w}^2\,\mathrm{d}x - \dot{C}_4 \qquad (48)
\end{aligned}
$$

by using the boundary conditions given by Eqs. (2) and (7). Solving Eq. (48) yields

$$
\dot{C}_4 \leqslant \rho R\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma - 2\rho\int_\Omega \dot{w}^2\,\mathrm{d}x. \qquad (49)
$$

The time derivative of crossing term C_2 can be written as

$$
\begin{aligned}
\dot{C}_2 &= \int_\Omega (f + P\Delta w)w\,\mathrm{d}x + \rho\int_\Omega \dot{w}^2\,\mathrm{d}x \\
&\leqslant -\left[P - \delta_3 m_1 - \frac{1}{2}cm_2\right]\int_\Omega |\nabla w|^2\,\mathrm{d}x + \rho\int_\Omega \dot{w}^2\,\mathrm{d}x \\
&\quad + \frac{1}{\delta_3}\int_\Omega f^2\,\mathrm{d}x + \frac{1}{2}c\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma \qquad (50)
\end{aligned}
$$

by using Eq. (3) and Eqs. (7)–(9). Substitution of Eqs. (42), (46), (47) and (49) into Eq. (16) yields

$$
\begin{aligned}
\dot{V} &\leqslant -\left\{c - \beta\left[R\left(\frac{c^2}{P} + \rho\right) + \frac{1}{2}c\right]\right\}\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma \\
&\quad - \beta\left[P - 2R\delta_2 - \delta_3 m_1 - \frac{1}{2}cm_2\right]\int_\Omega |\nabla w|^2\,\mathrm{d}x \\
&\quad - (\beta\rho - \delta_1)\int_\Omega \dot{w}^2\,\mathrm{d}x \\
&\quad + \left[\frac{1}{\delta_1} + \beta\left(\frac{2R}{\delta_2} + \frac{1}{\delta_3}\right)\right]\int_\Omega f^2\,\mathrm{d}x, \qquad (51)
\end{aligned}
$$

where for sufficiently small β, δ_1, δ_2, and δ_3,

$$
c \geqslant \beta\left[R\left(\frac{c^2}{P} + \rho\right) + \frac{1}{2}c\right], \qquad (52)
$$

$$
\varepsilon_1 = \beta\rho - \delta_1 > 0, \qquad (53)
$$

$$
\varepsilon_2 = \beta\left[P - 2R\delta_2 - \delta_3 m_1 - \frac{1}{2}cm_2\right] > 0, \qquad (54)
$$

$$
\varepsilon = \left(\frac{1}{\delta_1} + \frac{2\beta R}{\delta_2} + \frac{\beta}{\delta_3}\right)\max_{t\in[0,\infty)}\int_\Omega f^2\,\mathrm{d}x, \qquad (55)
$$

$$
\lambda_3 = \frac{\min(\varepsilon_1,\varepsilon_2)}{\max(\rho, P)} > 0. \qquad (56)
$$

Therefore, Eq. (35) holds and the response is weakly bounded. □

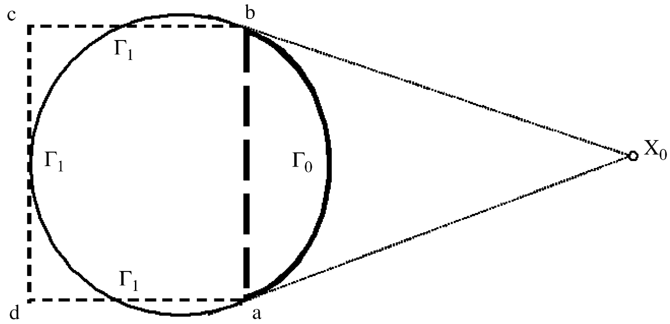

Fig. 3. Circular (solid) and rectangular (dashed) domain showing damped (thin) and undamped (thick) boundaries.

The partially damped boundary normal conditions given in Eqs. (39) and (40) require damping on part of the boundary ($\Gamma_1 \neq \emptyset$). We are free to choose $\mathbf{x}_0$ to determine the minimal Γ_1 for stability. If $\mathbf{x}_0$ is located at the center of a star shaped domain, then the entire boundary has $\mathbf{r} \cdot \mathbf{n} > 0$ so $\Gamma_1 = \Gamma$ and the entire boundary must be damped. Locating $\mathbf{x}_0$ outside Ω, however leads to $\Gamma_0 \neq \emptyset$ and part of the boundary need not be damped. In Fig. 3 are shown example circular (solid) and rectangular (dashed) domains with $\mathbf{x}_0 \notin \Omega$. In both cases, damping is not required on $\overline{ab}$. In the limit as $\mathbf{x}_0 \to \infty$, half of the circular domain is damped. For the rectangular domain as $\mathbf{x}_0 \to \infty$, $\mathbf{r} \cdot \mathbf{n} < 0$ on $\overline{ab}$ and $\mathbf{r} \cdot \mathbf{n} = 0$ on $\overline{bc}$ and $\overline{da}$, so $\mathbf{r} \cdot \mathbf{n} \leqslant 0$ on $\overline{da} \cup \overline{ab} \cup \overline{bc}$. Thus, only one side $\overline{cd} = \Gamma_1$ requires damping.

4. Distributed plates

In this section, we investigate the stability of distributed and boundary damped Kirchhoff plates with distributed excitation. We assume the plates are inextensible and homogeneous with uniform cross-section.

4.1. Distributed damped plates

The field equation of the distributed damped plate includes distributed viscous and material damping and forcing:

$$\rho \ddot{w} + b\dot{w} + D\Delta^2 \dot{w} + D_E \Delta^2 w = f \quad \text{in } \Omega \times R_+, \tag{57}$$

with boundary condition

$$w = 0 \quad \text{on } \Gamma \times R_+, \tag{58}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma \times R_+, \tag{59}$$

where D_E is the plate flexural rigidity. The initial conditions are given in Eqs. (12) and (13).

Theorem 3. *The response of the damped plate governed by Eqs.* (57)–(59) *is weakly bounded if either b or D is nonzero and $f \in \mathscr{L}_\infty$.*

Proof. The energy of the plate is

$$E = \frac{1}{2} \int_\Omega \left\{ 2(1-\mu) \left[\left(\frac{\partial^2 w}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 w}{\partial x_2^2} \right] + \rho \dot{w}^2 + D_E (\Delta w)^2 \right\} \mathrm{d}x, \tag{60}$$

where μ is Poisson's ratio. The Gaussian curvature $2(1-\mu)[(\partial^2 w/(\partial x_1 \partial x_2))^2 - (\partial^2 w/\partial x_1^2)(\partial^2 w/\partial x_2^2)]$ complicates the energy. For a clamped plate with either a rectangular domain or a smooth boundary, however, the Gaussian curvature integral is zero [12].

Elimination of the Gaussian curvature integral and differentiation of Eq. (60) produces

$$\dot{E} = \int_\Omega [\dot{w}(f - b\dot{w} - D\Delta^2\dot{w} - D_E\Delta^2 w) + D_E\Delta w\Delta\dot{w}]\,\mathrm{d}x$$
$$\leqslant -\left(b + \frac{D}{2m_1^2} - \delta_1\right)\int_\Omega \dot{w}^2\,\mathrm{d}x + \frac{1}{\delta_1}\int_\Omega f^2\,\mathrm{d}x - \frac{D}{2}\int_\Omega (\Delta\dot{w})^2\,\mathrm{d}x, \tag{61}$$

when Eqs. (2), (3), (7) and (8) are used.

Both viscous and material damping match the disturbance input, producing a negative kinetic energy term in $\dot{E}$. The energy cannot be used to prove stability, however, because the time derivative lacks the $-\int_\Omega (\Delta w)^2\,\mathrm{d}x$ term that is found in E. We therefore add the crossing term in Eq. (17) to form positive functional of Eq. (16). The crossing term can be bounded by Eq. (18), where

$$\eta = \frac{\rho\max(1, m_1^2)}{\min(\rho, D_E)}. \tag{62}$$

The time derivative of the crossing term given in Eq. (17) has the form of Eq. (26) with

$$\dot{C}_3 \leqslant D\delta_4 \int_\Omega (\Delta w)^2\,\mathrm{d}x + \frac{D}{\delta_4}\int_\Omega (\Delta\dot{w})^2\,\mathrm{d}x \tag{63}$$

and

$$\dot{C}_4 = -D_E \int_\Omega (\Delta w)^2\,\mathrm{d}x. \tag{64}$$

Substitution of Eqs. (22), (23), (63) and (64) into Eq. (26) produces

$$\dot{V} \leqslant -\left[b + \frac{D}{2m_1^2} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right)\right]\int_\Omega \dot{w}^2\,\mathrm{d}x$$
$$- \beta[D_E - (\delta_2 + b\delta_3)m_1^2 - D\delta_4]\int_\Omega (\Delta w)^2\,\mathrm{d}x$$
$$- D\left(\frac{1}{2} - \frac{\beta}{\delta_4}\right)\int_\Omega (\Delta\dot{w})^2\,\mathrm{d}x + \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\int_\Omega f^2\,\mathrm{d}x$$
$$\leqslant -\lambda_3 E + \varepsilon, \tag{65}$$

where, for sufficiently small β, δ_1, δ_2, δ_3, and δ_4,

$$\frac{1}{2} \geqslant \frac{\beta}{\delta_4}, \tag{66}$$

$$\varepsilon_1 = b + \frac{D}{2m_1^2} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right) > 0, \tag{67}$$

$$\varepsilon_2 = \beta[D_E - (\delta_2 + b\delta_3)m_1^2 - D\delta_4] > 0, \tag{68}$$

$$\varepsilon = \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\max_{t\in[0,\infty)}\int_\Omega f^2\,\mathrm{d}x < \infty, \tag{69}$$

$$\lambda_3 = \frac{\min(\varepsilon_1, \varepsilon_2)}{\max(\rho, D_E)} > 0. \tag{70}$$

Therefore, Eq. (35) holds and the system is weakly stable. □

4.2. Boundary damped plates

For the boundary clamped plate model, the viscous and material damping are removed from the field equation and clamped boundary condition is changed to a damper on Γ_1. The field equation and boundary conditions are:

$$\rho\ddot{w} + D_E\Delta^2 w = f \quad \text{in } \Omega \times R_+, \tag{71}$$

$$w = 0 \quad \text{on } \Gamma_0 \times R_+, \tag{72}$$

$$\frac{\partial}{\partial n} w = 0 \quad \text{on } \Gamma_0 \times R_+, \tag{73}$$

$$\Delta w = 0 \quad \text{on } \Gamma_1 \times R_+, \tag{74}$$

$$D_E \frac{\partial}{\partial n}\Delta w - c\dot{w} = 0 \quad \text{on } \Gamma_1 \times R_+, \tag{75}$$

and the initial conditions are given in Eqs. (12) and (13).

Theorem 4. *The response of the boundary damped plate governed by Eqs.* (71)–(75) *is bounded if* $c>0$, $f \in \mathscr{L}_\infty(\Omega)$, *and the normal boundary conditions given in Eqs.* (39) *and* (40) *are satisfied.*

Proof. The time derivative of the energy can be expressed as

$$\dot{E} \leqslant -c\int_{\Gamma_1} \dot{w}^2\,\mathrm{d}\Gamma + \delta_1\int_\Omega \dot{w}^2\,\mathrm{d}x + \frac{1}{\delta_1}\int_\Omega f^2\,\mathrm{d}x, \tag{76}$$

using the boundary conditions and Eq. (7).

The boundary damper does not match the distributed input, providing neither a negative kinetic nor potential energy domain integral term. A positive functional is defined as in Eq. (16) with a different crossing term

$$C(t) = \int_\Omega \rho\dot{w}(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x. \tag{77}$$

We can bound this crossing term with respect to the system energy as in Eq. (43) by using Eqs. (7), (8) and (44), where

$$\eta = \frac{\rho R \max(1, m_1)}{\min(\rho, D_E)}. \tag{78}$$

The time derivative of the crossing term can be expressed as

$$\begin{aligned}\dot{C} &= \int_\Omega (f - D_E\Delta^2 w)(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x + \int_\Omega \rho\dot{w}(\mathbf{r}\cdot\nabla\dot{w})\,\mathrm{d}x \\ &\leqslant Rm_1\delta_2\int_\Omega (\Delta w)^2\,\mathrm{d}x + \frac{R}{\delta_2}\int_\Omega f^2\,\mathrm{d}x + \dot{C}_3 + \dot{C}_4,\end{aligned} \tag{79}$$

by using Eqs. (2), (4), (7) and (8).

The third term in Eq. (79) simplifies as follows:

$$\begin{aligned}\dot{C}_3 &= -\int_\Omega D_E\Delta^2 w(\mathbf{r}\cdot\nabla w)\,\mathrm{d}x \\ &= -D_E\int_\Gamma \frac{\partial\Delta w}{\partial n}(\mathbf{r}\cdot\nabla w)\,\mathrm{d}\Gamma - D_E\int_\Omega \Delta(\mathbf{r}\cdot\nabla w)\Delta w\,\mathrm{d}x \\ &\quad + D_E\int_\Gamma \Delta w\frac{\partial}{\partial n}(\mathbf{r}\cdot\nabla w)\,\mathrm{d}\Gamma \\ &= -D_E\int_\Gamma \frac{\partial\Delta w}{\partial n}(\mathbf{r}\cdot\nabla w)\,\mathrm{d}\Gamma - \frac{1}{2}D_E\int_\Omega \mathbf{r}\cdot\nabla(\Delta w)^2\,\mathrm{d}x \\ &\quad + D_E\int_\Gamma \Delta w[\nabla(\mathbf{r}\cdot\nabla w)\cdot\mathbf{n}]\,\mathrm{d}\Gamma - 2D_E\int_\Omega (\Delta w)^2\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{Rc}{\delta_3}\int_{\Gamma_1} \dot{w}^2\,\mathrm{d}\Gamma + Rc\delta_3\int_{\Gamma_1}|\nabla w|^2\,\mathrm{d}\Gamma \\
&\quad + \frac{1}{2}D_E\int_{\Gamma_0}(\mathbf{r}\cdot\mathbf{n})(\Delta w)^2\,\mathrm{d}\Gamma - D_E\int_{\Omega}(\Delta w)^2\,\mathrm{d}x \\
&\leqslant -(D_E - Rcm_2\delta_3)\int_{\Omega}(\Delta w)^2\,\mathrm{d}x + \frac{Rc}{\delta_3}\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma,
\end{aligned} \tag{80}
$$

when using the boundary conditions and Eqs. (2)–(8). Based on the boundary conditions in Eq. (39), $\frac{1}{2}D_E\int_{\Gamma_0}(\mathbf{r}\cdot\mathbf{n})(\Delta w)^2\,\mathrm{d}\Gamma\leqslant 0$ can be dropped. The fourth term in Eq. (79) can be expressed as

$$
\begin{aligned}
\dot{C}_4 &= \int_{\Omega}\rho\dot{w}(\mathbf{r}\cdot\nabla\dot{w})\,\mathrm{d}x \\
&\leqslant \frac{\rho R}{2}\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma - \rho\int_{\Omega}\dot{w}^2\,\mathrm{d}x,
\end{aligned} \tag{81}
$$

by using the boundary conditions and Eqs. (2), (4) and (44).

Substitution of Eqs. (76), (79)–(81) into Eq. (16) produces

$$
\begin{aligned}
\dot{V} &\leqslant -\left[c - \beta R\left(\frac{\rho}{2}+\frac{c}{\delta_3}\right)\right]\int_{\Gamma_1}\dot{w}^2\,\mathrm{d}\Gamma \\
&\quad - (\beta\rho - \delta_1)\int_{\Omega}\dot{w}^2\,\mathrm{d}x + \left(\frac{1}{\delta_1}+\frac{\beta R}{\delta_2}\right)\int_{\Omega}f^2\,\mathrm{d}x \\
&\quad - \beta[D_E - R(m_1\delta_2 - cm_2\delta_3)]\int_{\Omega}(\Delta w)^2\,\mathrm{d}x \\
&\leqslant -\lambda_3 E + \varepsilon,
\end{aligned} \tag{82}
$$

where, for sufficiently small β, δ_1, δ_2, and δ_3,

$$c \geqslant \beta R\left(\frac{\rho}{2}+\frac{c}{\delta_3}\right), \tag{83}$$

$$\varepsilon_1 = \beta\rho - \delta_1 > 0, \tag{84}$$

$$\varepsilon_2 = \beta(D_E - Rm_1\delta_2 - Rcm_2\delta_3) > 0, \tag{85}$$

$$\varepsilon = \left(\frac{1}{\delta_1}+\frac{\beta R}{\delta_2}\right)\max_{t\in[0,\infty)}\int_{\Omega}f^2\,\mathrm{d}x < \infty, \tag{86}$$

$$\lambda_3 = \frac{\min(\varepsilon_1,\varepsilon_2)}{\max(\rho, D_E)} > 0. \tag{87}$$

Therefore, Eq. (35) holds and the response is weakly stable. □

5. Conclusions

In this paper it has been shown that distributed and boundary damping can ensure a bounded response for pinned membranes and clamped plates under distributed excitation. Either external, viscous damping or internal, material damping ensures weak stability with respect to the energy norm. The distributed input can include spatial and time variations provided it is $\mathscr{L}_2$ spatially and $\mathscr{L}_\infty$ temporally bounded, respectively. Thus, time-bounded point forces are allowed because they have a bounded $\mathscr{L}_2$ spatial norm. Boundary damping must satisfy the normal boundary conditions given in Eqs. (39) and (40) to ensure stability. Circular and rectangular domains satisfy these conditions with damping on half and one side, respectively. For each of the cases studied, $\varepsilon = 0$ if $f = 0$ so without inputs these systems are weakly exponentially stable.

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